Note: The MUST framework has 3 Components and each Component has Strands

Chapter 4

In spite of the dynamic nature of the teaching of mathematics, descriptions of knowledge, skills, and proficiencies for teaching mathematics often suggest they are static entities to be mastered,. As a consequence, the process of creating the framework for Mathematical Understanding for Secondary Teaching (MUST) has been a challenging task. The knowledge structure of competent mathematics teachers is much more complex than a specific list of the mathematics that teachers need to know and the mathematical processes that they need to be able to perform, and trying to articulate such a list is likely to be an unsuccessful venture. In contrast, we have worked, with the aid of many, to design a framework that showcases three important components of mathematical expertise that are useful in teaching mathematics at the secondary level. A lack of familiarity with a particular piece of mathematics can be overcome by a robust knowledge of other related pieces of mathematics. Good argumentation and discussion within a classroom can promote mathematical learning for both students and teachers. The framework we have built, with the help of the myriad mathematics educators, mathematicians, and teachers who participated at various points in the process, describes components of mathematical understanding that, we believe, need to be central to mathematics teacher education at both the preservice and inservice levels, because these components are useful to teachers as they continue to learn mathematics and continue to facilitate the learning of their students.

We have attempted to advance previous work, respond to critiques of earlier versions of our framework, and to remain faithful to our goal of relating the framework to actual classroom practice. Consequently, earlier versions (e.g., Mathematical Proficiency for Teaching) of the framework have been used in research and publications ( citations??), and are organized differently. Some ideas for advancement came from three conferences that were held to discuss the mathematical understanding that would be useful to teachers. The first conference was held in May 2007 at the Pennsylvania State University where we shared our goals, our research base, a subset of the situations we had developed, and struggles with categorizing mathematical understanding. The second conference was in March 2009 at the Pennsylvania State University where we shared our progress and sought specific feedback. The third conference was held in March 2010 at the University of Georgia where we discussed the Framework and the Situations with potential users. All of the conferences included mathematicians, mathematics educators, and teachers of mathematics at the secondary level. We expect that future work will lead to additional modifications as researchers and educators learn more about the mathematical understanding that is useful to mathematics teachers at the secondary level.

This chapter shares the evolution of the current MUST framework. We hope that the description of the process helps to explain the derivation of the three major components of Mathematical Proficiency, Mathematical Activity and Mathematical Work of Teaching, and encourages the reader to consider the multiple, overlapping dimensions of mathematical understanding for teaching at the secondary level.

**The First Component: Mathematical Proficiency**

The first component, that of Mathematical Proficiency, was a natural one with which to begin. The strands of proficiency that comprise that component were identified by the Mathematics Learning Study Committee that developed *Adding It Up: Helping Children Learn Mathematics*, and they were ones of importance and interest in developing a framework for mathematical understanding for secondary teaching. The strands of proficiency explained in *Adding It Up* were, however, designed for students in grades K through 8. Although, if students were expected to develop these proficiencies, it made sense that their teachers should also develop those proficiencies, we needed to determine how well those strands fit the mathematical understandings needed by teachers of secondary students. We expanded the set of the five strands of mathematical proficiency to include a sixth strand, historical and cultural knowledge. This latter strand had been one that was drawn to the attention of the Mathematics Learning Study Committee late in their work, and although it was the impression of members of the committee that historical and cultural knowledge should have been included as a strand. We proceeded to map the Situations to the (now) six strands of mathematical proficiency. We were able to exemplify each of the strands with multiple examples drawn from the Situations, providing us with evidence of a reasonable fit.

**PAT, DO WE NEED MORE HERE ABOUT THE DEVELOPMENT OF THE FIRST COMPONENT?**

**Development of the Second Component: Mathematical Activity**

Having produced, revised, vetted, and revised several times again more than 50 Situations, our goal was to use those Situations as data from which to develop a framework for Mathematical Understanding for Secondary Teaching. For the first component, we started with a known framework and exemplifying its strands by referencing the Situations we had created. We wondered whether important aspects of mathematical activity were missed in this effort. A second component was developed by starting with the Situations and extracting descriptions of the relevant mathematics from them. It seemed reasonable to start with the actions that were suggested in the Situations. Following are examples of mathematical actions of teaching that we sought a framework to account for:

1. *Creating a counterexample*. For example, use matrix operations as a counter to the claim that the associative property holds for multiplication of any mathematical entities. To accomplish this, teachers need to know the properties of the objects involved in the counterexample and variations of those properties.
2. *Creating an example or nonexample*. For example, a teacher may be creating a polynomial that factors over the real numbers while knowing to be careful if the degree of the polynomial is 4 or higher, or using the limit or derivative with a symbolic rule to create a graph that had particular pre-identified characteristics.
3. *Fitting a question in a larger setting in order to identify a special case of a broader category of mathematical objects*. For example, in a discussion of whether multiplication is commutative, embed the discussion in the case of matrices.
4. *Explaining why a process doesn’t generalize when trying to apply the process to a different entity*. For example, use the “students’ law of distributivity” (e.g., sin(*a* + *b*) = sin(*a*) + sin(*b*)) to demonstrate that the distributive property of multiplication over addition doesn’t necessarily generalize to distributing other functions over addition.

As we examined and classified actions involved in the mathematical tasks for teaching mathematics, we recognized their dependence on two phenomena: *mathematical* *actions* and *mathematical* *entities*. Any task for teaching mathematics can be characterized by combining actions with entities. Entities can be subdivided into smaller units. We came to think of the actions and entities as the nouns and verbs of the Situations. The lists of actions and entities that we identified in the Situations appear in Figure 1.

Over the course of our work on the Situations, we had generated additional lists of mathematical tasks of teaching secondary mathematics. For one of our conferences, we convened a group of experts in the area of mathematical knowledge for secondary teaching. They were people who had written curricula or expository pieces for teachers on the topic, and some of the discussion at that conference was directed toward describing mathematical tasks of teaching secondary mathematics. On another occasion, we had met in smaller groups to generate a list of ways in which teachers draw on their mathematical knowledge in the course of teaching secondary mathematics. Notes and transcripts from those meetings gave us a list of mathematical tasks. We sorted those tasks into the categories of actions shown in Figure 1. Although the tasks we classified were mathematical, the actions under which we classified them were not specifically mathematical. Using those mathematical tasks as an anchor, our goal was to develop a definition of each action if viewed as a mathematical action. The intention was to limit the definition for each of the actions so that they describe mathematical actions. For example, the verb, “recognize,” when used to describe a mathematical action can be limited to “*Recognizing mathematical properties, constraints, or structure in a given mathematical entity or setting, or across instances of a mathematical entity.”* We felt thatthese definitions of mathematical actions might be useful in helping us make progress toward a framework for Mathematical Understanding for Secondary Teaching that captured the mathematics of classroom-based Situations.

Our process of creating mathematically delimited definitions for the actions included procedures for validating our definitions. After having created a definition,

we held it up to each of the mathematical tasks of teaching we had classified under that action. When we were having difficulty making sense of what could be meant by a particular description of a mathematical task, we referred to meeting notes and lists from which we had drawn the task. Referring to the context in which the mathematical task of teaching was discussed enabled us to “clean up language” in the bullets. We felt that doing so could make the document more readable to others and could work to develop our own shared understanding of the actions and the tasks. It also allowed us to maintain consistency with what we perceived to be a reasonable interpretation of the meaning of a particular mathematical task.

Although our list contained most of the actions in Figure 1, we did not include some of the actions that appeared in the list. We did so for several reasons: meeting notes and lists did not support a mathematical definition of the action; the action seemed too broad to support a specific mathematical definition (e.g., “recall”); or it seemed to make sense to subsume the action within another action or fuse it with other actions (e.g., demonstrate/explain).

Following are the mathematically delimited definitions, followed by a list of mathematical actions that the definition was intended to capture and an example to illustrate the definition. Several of the actions we defined were additions to the original list.

**Recognize**

*Recognizing mathematical properties, constraints, or structure in a given mathematical entity or setting, or across instances of a mathematical entity*

* Recognizing an exhaustive set of cases
* Recognizing limitations of reasoning from diagrams
* Recognizing when a set of constraints defines a unique case and when it defines a set of cases
* Seeing and recognizing structure
* Having awareness of the importance of mathematical structure (set and operations on a set)
* Identifying special cases
* Distinguishing when certain properties hold and when they do not

EXAMPLE: Recognizing that strategic choices for pairwise groupings of numbers are critical to one way of developing the general formula for summing the first *n* natural numbers

**Choose**

*Considering and selecting from among known (to the one choosing) mathematical entities or settings based on known (to the one choosing) mathematical criteria*

* Choosing “good” examples and counterexamples; Knowing there are models for different types of mainly arithmetic operations
* Selecting a particular representation type that fits a given mathematical criterion
* Selecting models for given number types or operations (e.g., rational numbers, multiplication)
* Choosing a simpler problem on which to base claims by recognizing the structure of the problem or by recognizing similarity in problems
* Selecting problems to foreground a concept

EXAMPLE: The mathematical meaning of (for real numbers *a* and *b* and sometimes, but not always, with ) arises in several different mathematical settings, including: slope of a line, direct proportion, Cartesian product, factor pairs, and area of rectangles. One might choose slope of a line as a setting to illustrate the need for b≠0.

**Create**

*Creating a mathematical entity or setting from known (to the one creating) structure, constraints, or properties*

* Creating a counterexample for a given structure, constraint, or property
* Creating an example or non-example for a given structure, constraint, or property
* Creating equivalent equations to reveal information
* Creating problems to foreground a concept
* Creating a file whose creation requires mathematics beyond what the file is used to teach
* Constructing an object given a set of mathematical constraints
* Generating specific examples from an abstract idea
* Create a representation for a mathematical object with known structure, constraints, or properties

EXAMPLE: Constructing a special quadrilateral: Sketch quadrilateral ABCD with and such that ABCD is not a parallelogram.

**Use representations**

*For given representations, communicate about them and interpret them in the context of the signified, orchestrate movements between them, and craft analogies to describe the representations, objects, and relationships.*

* Translating among alternative representations (shifting representations) fluidly in order to foreground given attributes of the entity being represented
* Using/creating equivalent representations to reveal different information
* Using an analogy to accentuate selected characteristics of mathematical objects or relationships
* Using a variable to represent a quantifiable mathematical object
* Using geometric representations to give meaning to numerical or algebraic entities, and vice-versa.
* Developing and using analogies when appropriate to communicate with students at a range of levels
* Using mathematically precise language to communicate about a representation

EXAMPLE: Using tabular and graphical representations to estimate the value of 22.5

**Assess (interpret and adapt) the mathematics of the situation**
*Interpret and/or change certain mathematical conditions/constraints that are relevant to a mathematical activity*.

* Considering and/or analyzing special cases in mathematical contexts
* Knowing when relaxing mathematical constraints may be productive
* Considering what happens when certain mathematical conditions are not met
* Determining the mathematical conditions/constraints for which a statement is valid

EXAMPLE: Recognize the desirability of a modulus definition of absolute value in evaluating

EXAMPLE: It is not true that any number raised to the 0th power is equivalent to 1.

**Evaluate/Calculate**

**Extend**

*Extend domain, argument, or class of objects for which a mathematical statement is/remains valid*

* Structuring an argument so that it extends to a more general case
* Fitting a question in a larger arena
* Determining mathematical extensions to a given problem or question
* Recognizing mathematical relationships that allow one to extend a conclusion to a larger class
* Considering a definition in an expanded sense or altering the “universe” being considered
* Extending domain while preserving structure
* Looking at domain and extending domain

EXAMPLE: Extending the absolute value function from the real to the complex domain; extending the object "triangle" from Euclidean to spherical geometry

**Connect**

*By recognizing structural similarity, seek and make connections between (features of) representations of the same mathematical object or different methods for solving problems (e.g., Euclidean algorithm and long division algorithm because they are structurally similar), between mathematical objects of different classes, between objects in different systems, or between properties of an object in a different system*

* Connecting features of two representations of the same object
* Making connections among various “topics” in mathematics (e.g. connection between symbolic representation and analytic geometry)
* Looking for overarching mathematical ideas and/or structural similarity
* Making mathematical connections fluidly
* Recognizing concepts across different areas of mathematics as well as in the context of non-mathematical areas.
* Determining how two classes of objects are related structurally
* Seeking structural similarity between mathematical objects (i.e. similarities of Chinese remainder theorem and Lagrange interpolation)

EXAMPLE: Identifying structural similarities of the Euclidean algorithm and the long division algorithm

**Demonstrate and/or Explain**

*Demonstrate and/or explain mathematical operations, mathematical concepts, mathematical processes, or conventions through any of a range of representations (including physical actions)*

* Modeling a mathematical operation through physical actions
* Communicating mathematical thinking involved in problem solving
* Explaining mathematical phenomena in multiple ways–drawing on different perspectives
* Explaining the meaning of particular conventions in mathematical notation or language
* Making implied aspects of a problem explicit
* Explaining why a process does not generalize when trying to apply the process to a different entity
* Explaining the logic and/or organizing idea of a formal proof
* Using precise language

EXAMPLE: Distinguish between distance and displacement by having students walk around a room and quantify their displacement and quantify their distance traveled from their original position.

**Prove**

*Given a statement, formulate different levels and types of mathematically and pedagogically viable proofs.*

* Considering or generating empirical evidence and formulating conjectures, then trying to prove or disprove deductively
* Arguing by contradiction (excluded middle)
* Structuring a mathematical argument
* Stating assumptions on which a valid mathematical argument depends, and recognizing the need to do so
* Constructing proofs at an appropriate level

EXAMPLE: Arguing by contradiction (excluded middle): To prove that if the opposite angles of a quadrilateral are supplementary, then the quadrilateral can be inscribed in a circle, construct a circumcircle about three vertices of a quadrilateral and argue that if the fourth vertex can be in neither the interior nor the exterior or the circle, then the fourth vertex must be on the circumcircle, and therefore, the quadrilateral can be inscribed in a circle.

**Investigate**

*Take a mathematical action to find out more about the structure, constraints, and/or properties of a mathematical situation or a mathematical object.*

* Using technology to investigate a problem or estimate a value
* Solving unfamiliar problems or answering unexpected questions
* Formulating and reformulating conjectures

EXAMPLE: Is every polygon circumscribable?

**Apply**

*Employ algorithms, definitions, and technology in mathematical settings and/or real world quantitative settings when applicable.*

* Identifying which skills/algorithms/tools are important to a given mathematical situation and/or a real world context
* Utilizing algorithms/skills/tools when applicable

EXAMPLE: Using a dynagraph to explore ceiling and floor functions.

**Reason**

*Reason about a mathematical entity in one or more than one way, including, but not limited to: from mathematical definitions, from given conditionals, from and toward abstractions, by continuity, by analogy, and by using structurally equivalent statements.*

* Reasoning from mathematical definitions
* Thinking about a problem from different mathematical ways of reasoning
* Reasoning by continuity; reasoning from given conditionals; using logical equivalence (statement and contrapositive); using axiomatic structure
* Developing mathematically appropriate analogies
* Appealing to a definition
* Reasoning to account for the mathematics of a tool

EXAMPLE: Reasoning about the sum of the first n natural numbers by appealing to cases, by making strategic choices for pair-wise grouping of numbers, and by appealing to arithmetic sequences and properties of such sequences.

**Coordinate**

*Coordinate mathematical knowledge, student mathematical thinking, school curricula, and knowledge development.*

* Making associations between a representational approach and a problem type for typical problems in secondary mathematics
* Making connections “down” as well as “up” in the curriculum (e.g. commutative operations and non-communicative operations)
* Making connections between collegiate mathematics and school mathematics
* Making connections between development of student knowledge and curriculum
* Building connections - to curriculum, to students' thinking
* Identifying intermediate mathematics in the pursuit of a goal
* Situating particular mathematics (knowing where it can be applied, where it fits into the larger picture in mathematics)
* Analyzing how a particular curriculum supports the development of particular mathematical ideas
* Integrating 3 dimensions of mathematical idea development: (1) development of the idea within mathematics, (2) psychological – what someone already knows, what the conceptual obstacles are, and (3) development of the ideas within school mathematics – what is coming up and where have the students been
* Anticipating the kinds of mathematical questions or ideas a given curriculum or lesson might lead students to ask or develop
* Interpreting the mathematics in students’ questions or work
* Identifying the mathematical affordances and constraints that a particular curriculum sets up

EXAMPLE: Once students know that if , then or then they sometimes assume that if , then or . It would be useful for the teacher to recognize that students make this error and to understand the role of the absence of zero divisors (see integral domains in abstract algebra) in understanding why this is a misconception.

**Reflect (self-reflect)**

*Reflect on mathematical aspects of one’s practice or on one’s own doing math.*

* Reflecting on our own practice and on doing mathematics
* Recognizing one's own way of approaching mathematics (problem solving)

EXAMPLE: A teacher might reflect on decisions they make in the classrooms, attending to the mathematical concepts on which they draw.

**Knowledge and dispositions**

As we developed our definitions and examples of mathematical actions, we identified some (mental) actions that seemed to pertain to mathematical knowledge or dispositions toward mathematics. We separated those actions from the ones that seemed more indicative of mathematical activity. A list of those actions follows:

Knowledge:

* Knowing the idea of mathematical families and how to reason about them
* Knowing mathematical theories, extensions, and generalized systems
* Knowing how particular bits of mathematics fit into a larger body of mathematics
* Understanding the logic that underpins mathematics
* Understanding how mathematical thought progresses and how arguments are formed
* Knowing conditions under which certain properties of mathematical objects hold and when they do not
* Knowing that there are different ways of approaching mathematics problems
* Knowing how to interpret mathematical symbols; keeping the meaning of symbols in mind
* Knowing rules of reasoning in an axiomatic system (e.g. converse vs. contrapositive)

Dispositions:

* Having a view that mathematics develops and that significant mathematical ideas are developed
* Having a perspective that mathematics for teaching is an applied mathematics
* Designing mathematical lessons involves mathematical problem solving
* Appreciating alternative ways of solving mathematics problems

**The Second Component: Mathematical Activity**

While they seemed to capture well what we had seen in the Situations we wrote, the list of verbs, along with mathematically delimited definitions and actions captured by those verbs, did not seem broad enough to account for the inclusion of the Situations that might be included in the future. In an attempt to broaden our categories, we classified those we had identified into the four larger categories of mathematical noticing, mathematical reasoning, mathematical creating, and integrating strands of mathematical activity. We discovered that each of our previous categories mapped well to one these four übercategories, and that attempting to map the activity categories arising from our Situations work into these larger categories led us to uncover a missing subcategory (Defining), which seemed to fit well into the Mathematical Creating category. The following table shows the current category system as well as activities from our prior Situations work that map into those categories.

I. Mathematical noticing: *Recognize* **(***Recognize mathematical properties, constraints, or structure in a given mathematical entity or setting, or across instances of a mathematical entity) and choose* (*Consider and select from among known (to the one choosing) mathematical entities or settings based on known (to the one choosing) mathematical criteria)*

A. Structure of mathematical systems

B. Symbolic form

C. Form of an argument

 D. Connect within and outside mathematics

*Connect (By recognizing structural similarity, seek and make connections between (features of) representations of the same mathematical object or different methods for solving problems (e.g., Euclidean algorithm and long division algorithm because they are structurally similar), between mathematical objects of different classes, between objects in different systems, or between properties of an object in a different system.)*

II. Mathematical reasoning**:** *Reason* (*Reason about a mathematical entity in one or more than one way, including, but not limited to: from mathematical definitions, from given conditionals, from and toward abstractions, by continuity, by analogy, and by using structurally equivalent statements.)*

1. Justifying/proving

Prove **(***Given a statement, formulate different levels and types of mathematically and pedagogically viable proofs)*

1. Reasoning when conjecturing and generalizing

Investigate (*Take a mathematical action to find out more about the structure, constraints, and/or properties of a mathematical situation or a mathematical object)*

1. Constraining and extending

Extend (*Extend domain, argument, or class of objects for which a mathematical statement is/remains valid)*

Assess (interpret and adapt) the mathematics of the situation **(***Interpret and/or change certain mathematical conditions/constraints that are relevant to a mathematical activity)*

III. Mathematical creating*. Create (Creating a mathematical entity or setting from known (to the one creating) structure, constraints, or properties)*

1. Representing
2. Defining
3. Modifying/transforming/manipulating

Evaluate/Calculate

IV. Integrating strands of mathematical activity. *Coordinate (Coordinate mathematical knowledge, student mathematical thinking, school curricula, and knowledge development); Reflect (self-reflect) (Reflect on mathematical aspects of one’s practice or on one’s own doing math); and Apply**(Employ algorithms, definitions, and technology in mathematical settings and/or real world quantitative settings when applicable.)*

**Third Component: Mathematical Work of Teaching.**

The component entitled the Mathematical Work of Teaching evolved later in the process of designing a framework. Building on the Situations we had derived from our observations of the practice of teaching secondary mathematics, we had developed the first two components of the framework. These components were built on classroom-based situations, and used the nouns and verbs that arose in mathematical foci for those situations. We could see that these mathematical proficiencies and activities were useful in mathematics teaching at the secondary level, but we were still missing the description of the contexts in which aspects of mathematical understanding are particularly important for mathematics teaching.

As we pondered the missing component, we recalled a message from the first conference during which reviewers noted that we had not captured the work of teaching mathematics. Moreover, comments and reviews we received regarding the framework frequently fell within the space of pedagogy rather than within the space of mathematics. Although we constantly tried to focus on the mathematics of Mathematical Understanding for Secondary Teaching and to not get diverted into the space of pedagogical content knowledge, this technical distinction is often hard to capture. We decided that a needed component of our framework was to account for pedagogy that is designed to teach mathematics. We believed that these pedagogical components were particularly mathematical, and that this mathematical component of understanding for secondary teaching was missing from the first two components.

We turned our attention to the mathematical work of teaching, and thought of it as the *mathematical* understanding that helped teachers access, interpret, and build on the mathematical understanding of students. It is a component of mathematical understanding and is not pedagogical content knowledge. Teachers need a special kind of mathematical knowledge to analyze mathematical ideas that arise in the classroom. They need mathematical knowledge to ask the optimal question that will give them access to a student thinking and will help the student reassess his or her understanding. Teachers need to understand the mathematical connections within the curriculum and to understand how changing or supplementing the curriculum influences the development of mathematical ideas. They need mathematical understanding in order to assess the critical components of students' mathematical thinking. For example, an assessment related to the Pythagorean theorem should not be limited to whether or not a student can find the hypotenuse on a given right triangle. The teacher needs to understand mathematics well enough to assess whether or not students know when to apply the Pythagorean theorem, how it can be proved and under what conditions it can be used. Teachers need to have the mathematical understanding to reflect on what was accomplished in the lesson so that they can identify what mathematics needs to be embedded in tasks that will help students build on what they learned in previous lessons. This kind of mathematical understanding clearly overlaps with the other two components of mathematical understanding but it does focus on the mathematics understanding that will prepare a teacher to understand and enable someone else's learning of mathematics. Although many mathematical professions require communication of mathematics, they usually do not focus on advancing the mathematical understanding of a student. This is the responsibility of teachers. It is the work of teaching.

Although the strands of the Mathematical Work of Teaching were created through conference discussions, teachers’ experiences, and research literature, they were not a result of the analysis of the Situations, which was the origin for the other two components. However, many of the strands were observed in classroom events during which teachers engaged students in discussions. To understand the issues underlying the prompt for each of the Situations we wrote, a teacher must be able to analyze the mathematics that was involved. A teacher with mathematical understanding can access student thinking by raising questions related to the mathematical ideas presented in the foci of the situation. Teachers use their mathematical understandings daily to interpret the curriculum and assess students’ understanding of core ideas. This useful mathematical understanding is not typical of the work of other mathematical professions, but is critical mathematical understanding for teachers of mathematics. The Mathematical Work of Teaching component was created to capture the mathematical understanding that is needed to apply the other components of mathematical understanding to the teaching profession.

**Validation:**

The framework is built on incidents from classrooms, but the interpretation of the incidents and the organization of the components of Mathematics Understanding for Secondary Teaching were the products of our consultation with groups of mathematicians and other mathematics educators. It was important to know how the framework resonated with practicing mathematicians, secondary teachers, and other mathematics educators. Although valuable feedback was obtained at each conference and from individual reviewers, it was critical to try to use the framework with professionals who had not been involved in their development.

Several mathematicians have studied the situations and framework, . . .

**WE NEED SOME KIND Of SUMMARY FROM THE UGA CONFERENCE AND POST CONFERENCE.**

In a research study by Conner, Wilson and Kim (2011), practicing secondary teachers were given the prompts from classrooms and asked to identify the mathematics that would be useful for teachers who were faced with the opportunities that the prompt afforded. They were provided with only the prompt and were not given the foci or commentaries. They were asked to list mathematical knowledge and mathematical practices that would be critical for thinking about how to make the most of the learning opportunity. Their lists of important, useful mathematics were remarkably similar to the mathematical knowledge and practices identified by the authors of the situation foci and commentaries. In the same study, preservice teachers were asked to identify mathematical ideas they should know in order to be able to work with students or events that were described in given prompts, but they were not provided with the foci. Again, the preservice teachers provided many of the same mathematical ideas that were explained in the foci for the given prompts. This study suggests that the full Situations do focus on mathematical understanding that teachers recognize as important and useful for the work of teaching, and provides a type of validation for the mathematical understanding described in the MUST framework.

 As the situations were developed and the components of the framework were created, mathematicians and mathematics educators began to use the situations in their courses and presentations. The prompts are brief and quickly generated discussion. Students and audiences were captivated by the volume of mathematical ideas that were relevant to any given prompt. By adding a discussion of the foci in the situation or by generating new foci, educators were able to expose significant mathematical questions as well as understanding. We see this as different but equally valuable evidence that the situations are valuable vehicles for identifying components of mathematical understanding. Further examples are discussed in Chapter 5.

**ACTIONS:**

Recognize Refute Use

Recall Connect Apply

Order Simplify Investigate

Choose Create Evaluate

Classify Explain Demonstrate

Identify Extend Prove

**ENTITIES:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Mathematical objects** | **Representations** | **Statements** | **Deductive arguments** | **Connections** |
| NumbersShapesExpressionsPatternsFunctionsDiagramsDomains | VerbalNumericalAlgebraicGeometric | ExamplesNonexamplesDefinitionsConjecturesQuestionsConclusionsConditionsGeneral argumentsCounterexamples | AlgorithmsStrategiesConventional practicesInformal reasoningInductive argumentsProceduresDeductive arguments | Outside mathematics* Real world
* Other disciplines History and culture

Inside mathematics* Studied earlier,
* Studied alongside (connections to other concepts, methods, representations, etc.)
* Studied later
 |

*Figure 1.* Actions and entities identified in mathematical tasks for teaching secondary mathematics.